From linear resolutions to Cartwright-Sturmfels ideals

Aldo Conca

The goal of this talk is to trace the development from a result on linear resolutions to the identification of the notion of *Cartwright–Sturmfels ideals*. This journey highlights not only the motivation behind introducing these ideals but also their connections to other concepts in combinatorial commutative algebra such as universal Gröbner bases and binomial edge ideals. The talk is based on a series of papers coauthored with De Negri and Gorla, and a paper in preparation with De Negri and Welker. The starting point, however, is the following result from an older paper [CH] coauthored with Jürgen Herzog in 2003.

Theorem 1. *Let $S = K[x_1, \dots, x_n]$ be a standard graded polynomial ring over a field K, and let $V_1, \dots, V_m$ be vector spaces of linear forms. For each $i = 1, \dots, m$, let I_i be the ideal generated by V_i, and let $I = I_1 \cdots I_m$ be their product. Then I has a linear resolution.*

Since the powers of I are still products of ideals generated by linear forms, it follows that I^v has a linear resolution for all $v \in \mathbb{N}$, that is, I has *linear powers*. Back then, we already knew that having linear powers is often a consequence of "good" properties of the blow-up algebras associated with the ideal. In this context, the most natural object to study was the subalgebra

$$A = K[V_1 y_1, \dots, V_m y_m]$$

of the Segre product $B = K[x_j y_i : j \in [n],\ i \in [m]]$ and its "diagonal subalgebra"

$$C = K[V_1 \cdots V_m].$$

The defining equations of B are well known, and those of A can be obtained via an elimination process. This quickly led to the following conjecture, stated in [C] in 2007, whose main consequence was that the algebra C is Koszul.

Conjecture 2. *Let $R = K[t_{ij} : (i,j) \in [m] \times [n]]$, multigraded by $\deg(t_{ij}) = e_i \in \mathbb{Z}^m$. Then for every $g \in \mathrm{GL}_n(K)^m$ the ideal $g(I_2)$ has a Gröbner basis consisting of elements of degree $\leq (1, 1, \dots, 1)$ with respect to* all *term orders.*

Here I_2 is the ideal generated by the 2-minors of the matrix (t_{ij}), and g acts on R row-by-row, i.e., as a $\mathbb{Z}^m$-graded K-algebra automorphism.

In [C], I was able to prove the conjecture only in the case where g is either "generic" or a collection of permutation matrices. As a corollary I obtained a proof of (a version of) White conjecture in the case of transversal polymatroid. Later on, in 2010, Cartwright and Sturmfels [CS] proved the Conjecture **??** in general using my own result for that the conjecture holds for a generic g (!) along with certain considerations about multidegrees of multigraded ideals. These arguments involving multidegrees were new to me, and I set out to understand them better.

I pursued this understanding during my stay at MSRI in 2012, where I was also collaborating with De Negri and Gorla on a new proof of the Bernstein-Sturmfels-Zelevinsky theorem on universal Gröbner bases for maximal minors of generic matrices.

Suddenly, we realized that the two lines of inquiry were related. There was a common structural feature underlying both problems. This led us to define *Cartwright–Sturmfels ideals* as those multigraded ideals whose multigraded generic initial ideal is radical. This definition provided a unifying framework, led to a new, simple proof of the universal Gröbner basis theorem for maximal minors in a very general form.

Subsequent work revealed several further manifestations of Cartwright–Sturmfels ideals, particularly within combinatorial commutative algebra. Indeed new classes of Cartwright–Sturmfels ideals includes

(1) Generalized binomial edge ideals,

(2) Multiview ideals,

(3) Lovász-Saks-Schrijver ideals ideals of trees.

References

[CH] Conca, Herzog, *Castelnuovo-Mumford regularity of products of ideals*. Collect. Math. **54** (2003), no. 2, 137-152.

[C] Conca, *Linear spaces, transversal polymatroids and ASL domains*. J. Algebraic Combin. **25** (2007), no. 1, 25-41.

[CS] Cartwright, Sturmfels, *The Hilbert scheme of the diagonal in a product of projective spaces*. Int. Math. Res. Not. IMRN 2010, no. 9, 1741-1771.

[CDG] Conca, De Negri, Gorla, *Universal Gröbner Bases for Maximal Minors (2015), Multigraded Generic Initial Ideals of Determinantal Ideals (2017), Universal Gröbner Bases and Cartwright–Sturmfels Ideals (2018), Cartwright–Sturmfels Ideals Associated to Graphs and Linear Spaces (2018), Radical Generic Initial Ideals (2022), Radical Support for Multigraded Ideals (2023)*

[CDW] Conca, De Negri, Welker, *Generalized binomial edge ideals are Cartwright-Sturmfels* in preparation (2025).